Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators*

J. Carlos Díaz-Ramos   Eduardo García-Río

Ramón Vázquez-Lorenzo

Abstract

A complete description of Osserman four-manifolds whose Jacobi operators have a nonzero double root of the minimal polynomial is given.

1 Introduction

Geometric information about a pseudo-Riemannian manifold (M, g) is essentially encoded by the curvature tensor $R \in \otimes^4 T^*M$. Hence a central problem in differential geometry is to relate algebraic properties of the curvature tensor to the underlying geometry of the manifold. Due to the fact that the whole curvature tensor is so difficult to handle, the investigation usually focuses on different objects associated to the curvature tensor, the Jacobi operator being the most natural and widely investigated. A pseudo-Riemannian manifold (M, g) is said to be *Osserman* if the eigenvalues of the Jacobi operators are constant on the unit pseudo-sphere bundles $S^\pm(TM)$. Since the Ricci tensor is obtained by tracing the Jacobi operators, any Osserman metric is Einstein and, in particular, of constant sectional curvature in dimension 2, 3.

Dimension four is therefore the first non-trivial case for consideration in trying to determine the whole class of Osserman metrics. Moreover due to the connection between Osserman and (anti-) self-dual metrics many special features occur like the existence of *pointwise Osserman* metrics which are not Osserman, i.e., the eigenvalues of the Jacobi operators are still independent of the direction, but they may change from point to point. (See for example [12], [16], [19] and the references therein for more information on pointwise Osserman

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4-manifolds, and [5], [17], [18], [22], [23] for the higher dimensional Riemannian and pseudo-Riemannian cases).

It has been shown in [8] and [11] (see also [2]) that any four-dimensional Osserman metric is locally isometric to a two-point homogeneous space if the signature is either Riemannian or Lorentzian. However the situation is much more complicated for neutral metrics and there exist many examples of non-symmetric Osserman pseudo-Riemannian manifolds of neutral signature [13]. Indeed besides the partial results at [3] for Osserman $(+ + --)$-metrics with diagonalizable Jacobi operators and those at [15] for locally symmetric Osserman four-manifolds, the general problem of obtaining a complete description of four-dimensional Osserman metrics of neutral signature remains open.

For any non-null vector X in the $(+ + --)$-setting, the induced metric on $X^\perp$ is of Lorentzian signature, and hence the eigenvalue-structure does not completely characterize the Jacobi operators R_X. The consideration of the Jordan normal form led to introduce the so-called *Jordan-Osserman* metrics as those where the Jordan normal form of the Jacobi operators is constant on $S^\pm(TM)$. Four-dimensional Jordan-Osserman metrics were initially investigated by Blažić, Bokan and Rakić [3], who considered four different possibilities according to the behavior of the Jordan normal form of the Jacobi operators as follows:

(Ia): The Jacobi operators are diagonalizable, i.e.,

$$R_X = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}.$$

(Ib): The Jacobi operators have a complex eigenvalue, i.e.,

$$R_X = \begin{pmatrix} \alpha & -\beta & \\ \beta & \alpha & \\ & & \gamma \end{pmatrix}.$$

(II): There is a double root of the minimal polynomial of R_X, i.e.,

$$R_X = \begin{pmatrix} \alpha & & \\ & \beta & \\ & 1 & \beta \end{pmatrix}.$$

(III): There is a triple root of the minimal polynomial of R_X, i.e.,

$$R_X = \begin{pmatrix} \alpha & & \\ 1 & \alpha & \\ & 1 & \alpha \end{pmatrix}.$$

It is shown in [3] that Jordan-Osserman metrics with diagonalizable Jacobi operators are locally isometric to a real, complex or paracomplex space

form and also that Type Ib metrics cannot occur. Moreover, locally symmetric Jordan-Osserman $(+ + - -)$-metrics have diagonalizable Jacobi operators or they are isometric to some Type II metrics with nilpotent Jacobi operators [15]. The fact that all known examples of nonsymmetric Jordan-Osserman metrics had nilpotent Jacobi operators of degree two or three suggested that no other examples could exist (cf. [7], [12], [16]). However, very recently the authors have shown the existence of a family of Type II Jordan-Osserman metrics with non-nilpotent Jacobi operators [10]. The purpose of this work is to clarify the situation of Type II Jordan-Osserman metrics by proving the following

Main Theorem *Let (M, g) be a four-dimensional Type II Jordan-Osserman manifold. Then the Jacobi operators are two-step nilpotent or otherwise there exist local coordinates $(x_1, \ldots, x_4)$ where the metric is given by*

$$(1) \qquad dx^1 dx^3 + dx^2 dx^4 + \sum_{i \leq j = 3,4} s_{ij} dx^i dx^j$$

for some functions $s_{ij}(x_1, \ldots, x_4)$ as follows

$$(2) \qquad \begin{aligned} s_{33} &= x_1^2 \tfrac{\tau}{6} + x_1 P + x_2 Q + \tfrac{6}{\tau} \{Q(T-U) + V(P-V) - 2(Q_4 - V_3)\}, \\ s_{44} &= x_2^2 \tfrac{\tau}{6} + x_1 S + x_2 T + \tfrac{6}{\tau} \{S(P-V) + U(T-U) - 2(S_3 - U_4)\}, \\ s_{34} &= x_1 x_2 \tfrac{\tau}{6} + x_1 U + x_2 V + \tfrac{6}{\tau} \{-QS + UV + T_3 - U_3 + P_4 - V_4\}, \end{aligned}$$

and arbitrary functions P, Q, S, T, U, V depending only on the coordinates (x_3, x_4), where $\tau \neq 0$ denotes the scalar curvature.

The proof of the Main Theorem is based on the following facts.

Fact A. [1], [19] A four-dimensional pseudo-Riemannian manifold is pointwise Osserman if and only if it is Einstein self-dual (or anti-self-dual).

Fact B. [3, Corollary 8.3] A Type II Jordan-Osserman metric is Ricci flat (i.e., $\alpha = \beta = 0$) or otherwise $\alpha = 4\beta \neq 0$.

Fact C. [3, Proposition 8.4] A Type II Jordan-Osserman metric whose Jacobi operators are not nilpotent (i.e., $\alpha = 4\beta \neq 0$) admits a local parallel field of two-dimensional planes.

Therefore, we investigate Walker metrics (i.e., those admitting a locally defined two-dimensional degenerate parallel distribution) in detail in §2, with special attention to the (anti-) self-dual Weyl curvature tensors. A complete description of self-dual Walker metrics is given in §2.3. The integration of the Einstein equation for a self-dual Walker metric, which lets us determine all pointwise Osserman self-dual Walker metrics, is carried out in §3. This leads to (1)–(2), thus proving the Main Theorem.

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2 Four-dimensional Walker metrics

A *Walker manifold* is a triple $(M, g, \mathcal{D})$ where M is an n-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximum dimensionality ($r = \frac{n}{2}$). Since the dimension of a null plane is $r \leq \frac{n}{2}$, the lowest possible case of a Walker metric is that of $(++--)$-manifolds admitting a field of parallel null two-planes.

Observe that there is a tight connection between Walker structures and Osserman metrics. First of all, it is a matter of fact that all known examples of Osserman metrics with nondiagonalizable Jacobi operators (i.e., Types II and III) are realized as Walker metrics. On the other hand, Walker metrics appear as the underlying structure of several specific pseudo-Riemannian structures. For instance, the metric tensor of any para-Kähler [9] (and hence any hyper-symplectic [20]) structure is necessarily of Walker type. The same occurs for the underlying metric of real hypersurfaces in indefinite space forms whose shape operator is nilpotent [21].

2.1 The canonical form of a Walker metric

For our purposes it is convenient to use a special coordinate system associated to any Walker metric. So, let g be a four-dimensional pseudo-Riemannian metric admitting a two-dimensional parallel null distribution. A canonical form for such a metric has been obtained by Walker [24] showing the existence of suitable coordinates $(x_1, \ldots, x_4)$ where the metric expresses as

$$(3) \qquad g_{(x_1, x_2, x_3, x_4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}$$

for some functions a, b and c depending on the coordinates $(x_1, \ldots, x_4)$. As a matter of notation, throughout this work we denote by ∂_i the coordinate vectors, $i = 1, \ldots, 4$. Also, $h_{i_1 \ldots i_r}$ means partial derivatives $\frac{\partial h}{\partial x_{i_1} \ldots \partial x_{i_r}}$, for any function $h(x_1, \ldots, x_4)$. Now, a straightforward calculation from (3) shows that the Levi Civita connection is given by

$$\nabla_{\partial_1} \partial_3 = \tfrac{1}{2} a_1 \partial_1 + \tfrac{1}{2} c_1 \partial_2, \qquad \nabla_{\partial_1} \partial_4 = \tfrac{1}{2} c_1 \partial_1 + \tfrac{1}{2} b_1 \partial_2,$$

$$\nabla_{\partial_2} \partial_3 = \tfrac{1}{2} a_2 \partial_1 + \tfrac{1}{2} c_2 \partial_2, \qquad \nabla_{\partial_2} \partial_4 = \tfrac{1}{2} c_2 \partial_1 + \tfrac{1}{2} b_2 \partial_2,$$

$$(4)\, \nabla_{\partial_3} \partial_3 = \tfrac{1}{2}(aa_1 + ca_2 + a_3)\partial_1 + \tfrac{1}{2}(ca_1 + ba_2 - a_4 + 2c_3)\partial_2 - \tfrac{a_1}{2}\partial_3 - \tfrac{a_2}{2}\partial_4,$$

$$\nabla_{\partial_3} \partial_4 = \tfrac{1}{2}(a_4 + ac_1 + cc_2)\partial_1 + \tfrac{1}{2}(b_3 + cc_1 + bc_2)\partial_2 - \tfrac{c_1}{2}\partial_3 - \tfrac{c_2}{2}\partial_4,$$

$$\nabla_{\partial_4} \partial_4 = \tfrac{1}{2}(ab_1 + cb_2 - b_3 + 2c_4)\partial_1 + \tfrac{1}{2}(cb_1 + bb_2 + b_4)\partial_2 - \tfrac{b_1}{2}\partial_3 - \tfrac{b_2}{2}\partial_4,$$

and the Riemann curvature tensor $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ satisfies

$$R_{1313} = -\tfrac{1}{2}a_{11}, \quad R_{1314} = -\tfrac{1}{2}c_{11}, \quad R_{1323} = -\tfrac{1}{2}a_{12}, \quad R_{1324} = -\tfrac{1}{2}c_{12},$$

$$R_{1334} = \tfrac{1}{4}\left(-a_2b_1 + c_1c_2 + 2a_{14} - 2c_{13}\right),$$

$$R_{1414} = -\tfrac{1}{2}b_{11}, \quad R_{1423} = -\tfrac{1}{2}c_{12}, \quad R_{1424} = -\tfrac{1}{2}b_{12},$$

$$R_{1434} = \tfrac{1}{4}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 - 2b_{13} + 2c_{14}\right),$$

(5)

$$R_{2323} = -\tfrac{1}{2}a_{22}, \quad R_{2324} = -\tfrac{1}{2}c_{22},$$

$$R_{2334} = \tfrac{1}{4}\left(c_2^2 - a_2b_2 - a_1c_2 + a_2c_1 + 2a_{24} - 2c_{23}\right),$$

$$R_{2424} = -\tfrac{1}{2}b_{22}, \quad R_{2434} = \tfrac{1}{4}\left(a_2b_1 - c_1c_2 - 2b_{23} + 2c_{24}\right),$$

$$R_{3434} = \tfrac{1}{4}\big(-ac_1^2 - bc_2^2 + aa_1b_1 + ca_1b_2 - a_1b_3 + 2a_1c_4$$
$$+ca_2b_1 + ba_2b_2 + a_2b_4 + a_3b_1 - a_4b_2 - 2a_4c_1$$
$$+2b_2c_3 - 2b_3c_2 - 2cc_1c_2 - 2a_{44} - 2b_{33} + 4c_{34}\big).$$

Next, let $\rho(X, Y) = \operatorname{trace}\{U \rightsquigarrow R(X, U)Y\}$ and $\tau = \operatorname{trace} \rho$ be the Ricci tensor and the scalar curvature, respectively. Then

$$\rho_{13} = \tfrac{1}{2}\left(a_{11} + c_{12}\right), \quad \rho_{14} = \tfrac{1}{2}\left(b_{12} + c_{11}\right),$$

$$\rho_{23} = \tfrac{1}{2}\left(a_{12} + c_{22}\right), \quad \rho_{24} = \tfrac{1}{2}\left(b_{22} + c_{12}\right),$$

(6) $\quad \rho_{33} = \tfrac{1}{2}\left(-c_2^2 + a_1c_2 + a_2b_2 - a_2c_1 + aa_{11} + 2ca_{12} + ba_{22} + 2c_{23} - 2a_{24}\right),$

$$\rho_{34} = \tfrac{1}{2}\left(-a_2b_1 + c_1c_2 + a_{14} + b_{23} + ac_{11} + 2cc_{12} - c_{13} + bc_{22} - c_{24}\right),$$

$$\rho_{44} = \tfrac{1}{2}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 + ab_{11} + 2cb_{12} - 2b_{13} + bb_{22} + 2c_{14}\right),$$

and
(7)
$$\tau = a_{11} + b_{22} + 2c_{12}.$$

Further, let W denote the Weyl conformal curvature tensor given by

(8)
$$\begin{aligned}
W(X, Y, Z, T) \quad = \quad & R(X, Y, Z, T) \\
& + \tfrac{\tau}{(n-1)(n-2)}\{g(X, Z)g(Y, T) - g(Y, Z)g(X, T)\} \\
& - \tfrac{1}{n-2}\{\rho(X, Z)g(Y, T) - \rho(Y, Z)g(X, T) \\
& \qquad + \rho(Y, T)g(X, Z) - \rho(X, T)g(Y, Z)\}.
\end{aligned}$$

2.2 Self-duality and anti-self-duality conditions

Considering the curvature tensor R as an endomorphism of $\Lambda^2(M)$, we have the following $O(2, 2)$-decomposition

(9)
$$R \equiv \tfrac{\tau}{12}\, Id_{\Lambda^2} + \rho_0 + W : \; \Lambda^2 \to \Lambda^2$$

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where ρ_0 denotes the traceless Ricci tensor, $\rho_0(X,Y) = \rho(X,Y) - \frac{\tau}{4} g(X,Y)$. The Hodge star operator $* : \Lambda^2 \to \Lambda^2$ associated to any $(++--)$-metric induces a further splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, where $\Lambda^2_\pm$ denotes the ± 1-eigenspaces of the Hodge star operator, that is, $\Lambda^2_\pm = \{\alpha \in \Lambda^2(M)/ * \alpha = \pm \alpha\}$. Correspondingly, the curvature tensor further decomposes as

$$(10) \qquad R \equiv \frac{\tau}{12} \, Id_{\Lambda^2} + \rho_0 + W^+ + W^-,$$

where $W^\pm = \frac{1}{2}(W \pm *W)$. Recall that a pseudo-Riemannian 4-manifold is called *self-dual* (resp., *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$). The connection between Osserman and (anti-) self-dual metrics relies on Fact A, and thus the analysis of the (anti-) self-duality conditions will play a basic role in what follows.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis and, as a convention, assume that e_1 and e_2 are spacelike while e_3 and e_4 are timelike vectors. Now, local bases of the spaces of self-dual and anti-self-dual two-forms can be constructed as

$$\Lambda^2_\pm = \langle \{E^\pm_1, E^\pm_2, E^\pm_3\} \rangle,$$

where

$$E^\pm_1 = \frac{e^1 \wedge e^2 \pm e^3 \wedge e^4}{\sqrt{2}}, \quad E^\pm_2 = \frac{e^1 \wedge e^3 \pm e^2 \wedge e^4}{\sqrt{2}}, \quad E^\pm_3 = \frac{e^1 \wedge e^4 \mp e^2 \wedge e^3}{\sqrt{2}}.$$

Here observe that the Hodge star operator satisfies

$$e^i \wedge e^j \wedge \star(e^k \wedge e^l) = (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \, \varepsilon_i \varepsilon_j \, e^1 \wedge e^2 \wedge e^3 \wedge e^4,$$

where $\varepsilon_i = g(e_i, e_i)$. Further note that $\langle E^\pm_1, E^\pm_1 \rangle = 1$, $\langle E^\pm_2, E^\pm_2 \rangle = -1$, $\langle E^\pm_3, E^\pm_3 \rangle = -1$, and therefore with respect to the above bases the self-dual and anti-self-dual Weyl curvature operators $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ have the following matrix form:

$$(11) \qquad W^\pm = \begin{pmatrix} W^\pm_{11} & W^\pm_{12} & W^\pm_{13} \\ -W^\pm_{12} & -W^\pm_{22} & -W^\pm_{23} \\ -W^\pm_{13} & -W^\pm_{23} & -W^\pm_{33} \end{pmatrix},$$

where $W^\pm_{ij} = W(E^\pm_i, E^\pm_j)$ and $W(e^i \wedge e^j, e^k \wedge e^l) = W(e_i, e_j, e_k, e_l)$.

Next, for a Walker metric (3) an orthonormal basis can be specialized by using the canonical coordinates as follows:

$$(12) \qquad \begin{aligned} e_1 &= \tfrac{1}{2}(1-a)\partial_1 + \partial_3, & e_2 &= -c\partial_1 + \tfrac{1}{2}(1-b)\partial_2 + \partial_4, \\ e_3 &= -\tfrac{1}{2}(1+a)\partial_1 + \partial_3, & e_4 &= -c\partial_1 - \tfrac{1}{2}(1+b)\partial_2 + \partial_4. \end{aligned}$$

Now, a long but straightforward calculation using (5), (6) and (7) shows that

the components of W^- in (11) are given by

$$W_{11}^- = -\tfrac{1}{12}(a_{11} + 3a_{22} + 3b_{11} + b_{22} - 4c_{12}),$$
$$W_{22}^- = -\tfrac{1}{6}(a_{11} + b_{22} - 4c_{12}),$$
(13)
$$W_{33}^- = \tfrac{1}{12}(a_{11} - 3a_{22} - 3b_{11} + b_{22} - 4c_{12}),$$
$$W_{12}^- = \tfrac{1}{4}(a_{12} + b_{12} - c_{11} - c_{22}),$$
$$W_{13}^- = \tfrac{1}{4}(a_{22} - b_{11}),$$
$$W_{23}^- = -\tfrac{1}{4}(a_{12} - b_{12} + c_{11} - c_{22}),$$

while the components of W^+ are determined by W_{11}^+, W_{12}^+ and the scalar curvature as follows:

(14) $\quad W_{22}^+ = -\dfrac{\tau}{6}, \qquad W_{33}^+ = W_{11}^+ + \dfrac{\tau}{6}, \qquad W_{13}^+ = W_{11}^+ + \dfrac{\tau}{12}, \qquad W_{23}^+ = W_{12}^+,$

the expressions of W_{11}^+ and W_{12}^+ being

$$W_{11}^+ = \tfrac{1}{12}(6ca_1b_2 - 6a_1b_3 - 6ba_1c_2 + 12a_1c_4 - 6ca_2b_1 + 6a_2b_4 + 6ba_2c_1$$
$$\quad + 6a_3b_1 - 6a_4b_2 - 12a_4c_1 + 6ab_1c_2 - 6ab_2c_1 + 12b_2c_3 - 12b_3c_2$$
(15)
$$\quad - a_{11} - 12c^2a_{11} - 12bca_{12} + 24ca_{14} - 3b^2a_{22} + 12ba_{24} - 12a_{44}$$
$$\quad - 3a^2b_{11} + 12ab_{13} - b_{22} - 12b_{33} + 12acc_{11} - 2c_{12} + 6abc_{12}$$
$$\quad - 24cc_{13} - 12ac_{14} - 12bc_{23} + 24c_{34}),$$

(16)
$$W_{12}^+ = \tfrac{1}{4}(-2ca_{11} - ba_{12} + 2a_{14} + ab_{12} - 2b_{23} + ac_{11} - 2cc_{12} - 2c_{13}$$
$$\quad - bc_{22} + 2c_{24}).$$

Remark 1 It is important to recognize here that the connection between Einstein (anti-) self-dual and pointwise Osserman manifolds at Fact A goes further to the Jordan normal forms of the nonzero part of the Weyl curvature tensor $W^{\pm}$ and the Jacobi operators (cf. [12]). So pointwise Osserman manifolds whose Jacobi operators are of Type Ia, Ib, II or III correspond to self-dual (or anti-self-dual) Einstein manifolds whose self-dual (or anti-self-dual) Weyl curvature tensor is of Type Ia, Ib, II or III, respectively (see also [4]).

The relations in (14) among the components of W^+ show that

$$W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \tfrac{\tau}{12} \\ -W_{12}^+ & \tfrac{\tau}{6} & -W_{12}^+ \\ -(W_{11}^+ + \tfrac{\tau}{12}) & -W_{12}^+ & -(W_{11}^+ + \tfrac{\tau}{6}) \end{pmatrix},$$

and, as a consequence, the eigenvalues of W^+ are given by

(17)
$$\left\{ \frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12} \right\}.$$

Since the induced metric on Λ_+^2 is of Lorentzian signature the structure of W^+ is determined by its Jordan normal form, which may correspond to Type Ia or type II/III, depending on whether W^+ is diagonalizable or not. A straightforward calculation shows that

$$\left(W^+ - \frac{\tau}{6}Id\right) \cdot \left(W^+ + \frac{\tau}{12}Id\right) = \frac{\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2}{48} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

from where we have the following:

(i) If $\tau \neq 0$, then W^+ has nonzero eigenvalues $\left\{\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}\right\}$, and

(18) $$\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0$$

is the necessary and sufficient condition for the diagonalizability of W^+. If (18) does not hold, then $-\frac{\tau}{12}$ is a double root of the minimal polynomial of W^+.

(ii) If $\tau = 0$, then W^+ vanishes if and only if $W_{11}^+ = W_{12}^+ = 0$ and moreover

(ii.1) W^+ is two-step nilpotent if and only if $W_{11}^+ \neq 0$ and $W_{12}^+ = 0$,

(ii.2) W^+ is three-step nilpotent if and only if $W_{12}^+ \neq 0$.

On the other hand, observe from (17) that any anti-self-dual Walker metric has vanishing scalar curvature and hence Einstein anti-self-dual Walker metrics are Ricci flat.

2.3 Explicit form of Self-dual Walker metrics

Recall that our main purpose is to obtain a description of non Ricci flat Type II Jordan-Osserman four-manifolds. Then, as a consequence of Remark 1, we restrict our analysis to self-dual Walker metrics. Thus, in what remains of this section we will give a complete description of self-dual Walker metrics by integrating the PDE system given by (13).

Theorem 2 *A Walker metric (3) is self-dual if and only if the defining functions $a(x_1, x_2, x_3, x_4)$, $b(x_1, x_2, x_3, x_4)$ and $c(x_1, x_2, x_3, x_4)$ are given by*

$$a(x_1, x_2, x_3, x_4) = x_1^3\mathcal{A} + x_1^2\mathcal{B} + x_1^2 x_2\mathcal{C} + x_1 x_2\mathcal{D} + x_1 P + x_2 Q + \xi,$$

$$b(x_1, x_2, x_3, x_4) = x_2^3\mathcal{C} + x_2^2\mathcal{E} + x_1 x_2^2\mathcal{A} + x_1 x_2\mathcal{F} + x_1 S + x_2 T + \eta,$$

(19)

$$c(x_1, x_2, x_3, x_4) = \tfrac{1}{2}x_1^2\mathcal{F} + \tfrac{1}{2}x_2^2\mathcal{D} + x_1^2 x_2\mathcal{A} + x_1 x_2^2\mathcal{C} + \tfrac{1}{2}x_1 x_2(\mathcal{B} + \mathcal{E})$$

$$+ x_1 U + x_2 V + \gamma,$$

where capital, calligraphic and Greek letters are all arbitrary smooth functions depending only on the coordinates (x_3, x_4).

Proof. First of all observe that using (13) the self-duality can be initially characterized by means of the following five PDEs:

$$a_{22} = 0,$$

$$b_{11} = 0,$$

(20)
$$a_{12} - c_{22} = 0,$$

$$b_{12} - c_{11} = 0,$$

$$a_{11} + b_{22} - 4c_{12} = 0.$$

So (19) is obtained as the solution of (20). We proceed.

The first step. First and second equations in (20) translate into

(21)
$$a(x_1, x_2, x_3, x_4) = x_2 A(x_1, x_3, x_4) + B(x_1, x_3, x_4),$$
$$b(x_1, x_2, x_3, x_4) = x_1 C(x_2, x_3, x_4) + D(x_2, x_3, x_4)$$

and hence the third equation in (20), i.e., $c_{22}(x_1, x_2, x_3, x_4) = A_1(x_1, x_3, x_4)$, implies that

$$(22)\ c(x_1, x_2, x_3, x_4) = \frac{1}{2} x_2^2 A_1(x_1, x_3, x_4) + x_2 E(x_1, x_3, x_4) + F(x_1, x_3, x_4).$$

The second step. In this step we make a long but straightforward use of the fourth equation in (20), which by (21) and (22) means

(23)
$$\frac{1}{2} x_2^2 A_{111}(x_1, x_3, x_4) + x_2 E_{11}(x_1, x_3, x_4)$$
$$+ F_{11}(x_1, x_3, x_4) - C_2(x_2, x_3, x_4) = 0.$$

First, appropriate successive differentiations in the above equation lead to $A_{1111}(x_1, x_3, x_4) = 0$, $E_{111}(x_1, x_3, x_4) = 0$ and $F_{111}(x_1, x_3, x_4) = 0$, and hence

$$A(x_1, x_3, x_4) = x_1^3 G(x_3, x_4) + x_1^2 \mathcal{C}(x_3, x_4) + x_1 \mathcal{D}(x_3, x_4) + Q(x_3, x_4),$$

(24) $$E(x_1, x_3, x_4) = x_1^2 H(x_3, x_4) + x_1 I(x_3, x_4) + V(x_3, x_4),$$

$$F(x_1, x_3, x_4) = x_1^2 J(x_3, x_4) + x_1 U(x_3, x_4) + \gamma(x_3, x_4).$$

By (24), condition (23) reduces to

(25) $$3x_2^2 G(x_3, x_4) + 2x_2 H(x_3, x_4) + 2J(x_3, x_4) - C_2(x_2, x_3, x_4) = 0,$$

from where $C_{222}(x_2, x_3, x_4) = 6G(x_3, x_4)$, which implies that

(26) $$C(x_2, x_3, x_4) = x_2^3 G(x_3, x_4) + x_2^2 \mathcal{A}(x_3, x_4) + x_2 \mathcal{F}(x_3, x_4) + S(x_3, x_4).$$

The final form of (25) is given by

$$2x_2(\mathcal{A}(x_3, x_4) - H(x_3, x_4)) - 2J(x_3, x_4) + \mathcal{F}(x_3, x_4) = 0,$$

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and as a consequence

$$(27) \qquad H(x_3, x_4) = \mathcal{A}(x_3, x_4), \qquad J(x_3, x_4) = \frac{1}{2}\mathcal{F}(x_3, x_4).$$

Collecting together the information in (24), (26) and (27), at the end of this step we have that (21) and (22) transform into

$$a(x_1, x_2, x_3, x_4) = x_1^3 x_2 G + x_1^2 x_2 \mathcal{C} + x_1 x_2 \mathcal{D} + x_2 Q + B(x_1, x_3, x_4),$$

$$b(x_1, x_2, x_3, x_4) = x_1 x_2^3 G + x_1 x_2^2 \mathcal{A} + x_1 x_2 \mathcal{F} + x_1 S + D(x_2, x_3, x_4),$$

$$(28) \quad c(x_1, x_2, x_3, x_4) = \tfrac{3}{2} x_1^2 x_2^2 G + \tfrac{1}{2} x_1^2 \mathcal{F} + \tfrac{1}{2} x_2^2 \mathcal{D} + x_1^2 x_2 \mathcal{A} + x_1 x_2^2 \mathcal{C}$$

$$+ x_1 x_2 I + x_1 U + x_2 V + \gamma.$$

The third step. To finish the determination of the defining functions we deal with the last equation in (20), which under the expressions in (28) and differentiating by x_1 and x_2 leads to

$$(29) \qquad\qquad G(x_3, x_4) = 0,$$

and hence that equation reduces to

$$(30) \quad \begin{aligned} 6x_1 \mathcal{A}(x_3, x_4) + 6x_2 \mathcal{C}(x_3, x_4) + 4I(x_3, x_4) \\ - B_{11}(x_1, x_3, x_4) - D_{22}(x_2, x_3, x_4) = 0. \end{aligned}$$

It follows that $B_{1111}(x_1, x_3, x_4) = 0$, from where

$$(31) \quad B(x_1, x_3, x_4) = x_1^3 L(x_3, x_4) + x_1^2 \mathcal{B}(x_3, x_4) + x_1 P(x_3, x_4) + \xi(x_3, x_4)$$

and (30) takes the form

$$(32) \quad \begin{aligned} 6x_1(\mathcal{A}(x_3, x_4) - L(x_3, x_4)) + 6x_2 \mathcal{C}(x_3, x_4) \\ + 4I(x_3, x_4) - 2\mathcal{B}(x_3, x_4) - D_{22}(x_2, x_3, x_4) = 0, \end{aligned}$$

which leads to

$$(33) \qquad\qquad L(x_3, x_4) = \mathcal{A}(x_3, x_4).$$

Thus, by (29), (31) and (33), the expression of a in (28) transforms into

$$(34) \quad a(x_1, x_2, x_3, x_4) = x_1^3 \mathcal{A} + x_1^2 \mathcal{B} + x_1^2 x_2 \mathcal{C} + x_1 x_2 \mathcal{D} + x_1 P + x_2 Q + \xi,$$

which finishes the process for this defining function. At this point, (32) has the form

$$(35) \quad 6x_2 \mathcal{C}(x_3, x_4) + 4I(x_3, x_4) - 2\mathcal{B}(x_3, x_4) - D_{22}(x_2, x_3, x_4) = 0,$$

from where $D_{2222}(x_2, x_3, x_4) = 0$, and hence

$$(36) \quad D(x_2, x_3, x_4) = x_2^3 M(x_3, x_4) + x_2^2 \mathcal{E}(x_3, x_4) + x_2 T(x_3, x_4) + \eta(x_3, x_4).$$

Then, (35) reduces to

$$3x_2(\mathcal{C}(x_3, x_4) - M(x_3, x_4)) + 2I(x_3, x_4) - \mathcal{B}(x_3, x_4) - \mathcal{E}(x_3, x_4) = 0,$$

so we conclude that

(37)
$$M(x_3, x_4) = \mathcal{C}(x_3, x_4), \qquad I(x_3, x_4) = \frac{1}{2}(\mathcal{B} + \mathcal{E}).$$

Thus, (29), (36) and (37) show that b and c in (28) take the desired form

$$b(x_1, x_2, x_3, x_4) = x_2^3 \mathcal{C} + x_2^2 \mathcal{E} + x_1 x_2^2 \mathcal{A} + x_1 x_2 \mathcal{F} + x_1 S + x_2 T + \eta,$$

(38) $\quad c(x_1, x_2, x_3, x_4) = \frac{1}{2}x_1^2 \mathcal{F} + \frac{1}{2}x_2^2 \mathcal{D} + x_1^2 x_2 \mathcal{A} + x_1 x_2^2 \mathcal{C} + \frac{1}{2}x_1 x_2(\mathcal{B} + \mathcal{E})$

$$+ x_1 U + x_2 V + \gamma,$$

finishing the proof. $\qquad\qquad\square$

Remark 3 A four-dimensional Walker metric is said to be *strict* if it admits two orthogonal parallel null vector fields rather than a parallel two-dimensional null distribution. It follows from the work of Walker that any strict Walker metric is given by (3) for any functions a, b and c depending only on the coordinates (x_3, x_4). Now (6) and Theorem 2 imply that any strict Walker metric is Ricci flat and self-dual, and hence Osserman. Moreover, Remark 1 shows that the Jacobi operators are identically zero or otherwise they are two-step nilpotent (depending on whether $W_{11}^+ = 2c_{34} - a_{44} - b_{33}$ vanishes or not).

3 Proof of the Main Theorem

Recall that our purpose is to obtain a local description of Type II Jordan-Osserman metrics whose Jacobi operators have nonzero eigenvalues. In such a case, the eigenvalues must be in a ratio $1 : \frac{1}{4}$ and the underlying metric is a Walker metric (cf. Facts B and C). Therefore in order to achieve the desired result, only Osserman metrics on Walker manifolds deserve further consideration and moreover, it immediately follows from Remark 1 that we may restrict to those being self-dual. In what follows we obtain a complete description of self-dual Einstein Walker metrics in Theorem 4, from where our main result is derived.

Theorem 4 *A Walker metric* (3) *is pointwise Osserman self-dual if and only if one of the following holds:*

(i) *The scalar curvature τ is nonzero and the metric tensor is completely determined by the functions $a(x_1, \ldots, x_4)$, $b(x_1, \ldots, x_4)$ and $c(x_1, \ldots, x_4)$ as follows*

$$a = x_1^2 \frac{\tau}{6} + x_1 P + x_2 Q + \frac{6}{\tau}\left\{Q(T - U) + V(P - V) - 2(Q_4 - V_3)\right\},$$

(39) $\quad b = x_2^2 \frac{\tau}{6} + x_1 S + x_2 T + \frac{6}{\tau}\left\{S(P - V) + U(T - U) - 2(S_3 - U_4)\right\},$

$$c = x_1 x_2 \frac{\tau}{6} + x_1 U + x_2 V + \frac{6}{\tau}\left\{-QS + UV + T_3 - U_3 + P_4 - V_4\right\},$$

where capital letters are arbitrary functions depending only on (x_3, x_4).

The Jacobi operators have eigenvalues $\{0, \frac{\tau}{6}, \frac{\tau}{24}, \frac{\tau}{24}\}$ and they are diagonalizable if and only if (18) holds. Otherwise, $\frac{\tau}{24}$ is a double root of the minimal polynomial of the Jacobi operators and the Walker manifold is Jordan-Osserman on the open set where (18) does not hold.

(ii) *The scalar curvature vanishes and the metric tensor is given by*

(40)
$$a = x_1 P + x_2 Q + \xi,$$
$$b = x_1 S + x_2 T + \eta,$$
$$c = x_1 U + x_2 V + \gamma,$$

where P, Q, S, T, U, V, ξ, η and γ are smooth functions depending only on (x_3, x_4) satisfying

(41)
$$2(Q_4 - V_3) = Q(T - U) + V(P - V),$$
$$2(S_3 - U_4) = S(P - V) + U(T - U),$$
$$T_3 - U_3 + P_4 - V_4 = QS - UV.$$

In this case, the Jacobi operators have zero eigenvalues and their minimal polynomials satisfy:

(ii.1) *The Jacobi operators are vanishing (i.e., Type Ia) if and only if*

(42)
$$T_3 + U_3 - P_4 - V_4 = 0$$

and W_{11}^+ given by (15) is also null.

(ii.2) *The Jacobi operators are two-step nilpotent (i.e., Type II) if and only if (42) holds and W_{11}^+ given by (15) is nonnull.*

(ii.3) *The Jacobi operators are three-step nilpotent (i.e. Type III) if and only if (42) does not hold.*

Proof. Let $\rho_0 = \rho - \frac{\tau}{4} g$ be the trace-free Ricci tensor. Then the Einstein equations for a general Walker metric (3) are as follows.

$$(\rho_0)_{13} = -(\rho_0)_{24} = (\rho_0)_{31} = -(\rho_0)_{42} = \tfrac{1}{4}\left(a_{11} - b_{22}\right) = 0,$$

$$(\rho_0)_{14} = (\rho_0)_{41} = \tfrac{1}{2}\left(b_{12} + c_{11}\right) = 0,$$

$$(\rho_0)_{23} = (\rho_0)_{32} = \tfrac{1}{2}\left(a_{12} + c_{22}\right) = 0,$$

$$(\rho_0)_{33} = \tfrac{1}{4}\big(2a_1c_2 + 2a_2b_2 - 2a_2c_1 - 2c_2^2 + aa_{11}$$

(43)
$$+4ca_{12} + 2ba_{22} - 4a_{24} - ab_{22} - 2ac_{12} + 4c_{23}\big) = 0,$$

$$(\rho_0)_{34} = (\rho_0)_{43} = \tfrac{1}{4}\big(-2a_2b_1 + 2c_1c_2 - ca_{11} + 2a_{14} - cb_{22}$$

$$+2b_{23} + 2ac_{11} + 2cc_{12} - 2c_{13} + 2bc_{22} - 2c_{24}\big) = 0,$$

$$(\rho_0)_{44} = \tfrac{1}{4}\big(2a_1b_1 - 2b_1c_2 + 2b_2c_1 - 2c_1^2 - ba_{11}$$

$$+2ab_{11} + 4cb_{12} - 4b_{13} + bb_{22} - 2bc_{12} + 4c_{14}\big) = 0.$$

Now, since the manifold is self-dual, the defining functions are completely determined by (19) (cf. Theorem 2). Then, computing the first three equations in (43), we get

$$2x_1\mathcal{A} - 2x_2\mathcal{C} + \mathcal{B} - \mathcal{E} = 0, \qquad 2x_2\mathcal{A} + \mathcal{F} = 0, \qquad 2x_1\mathcal{C} + \mathcal{D} = 0,$$

from where
(44)
$$\mathcal{A} = \mathcal{C} = \mathcal{D} = \mathcal{F} = 0, \qquad \mathcal{E} = \mathcal{B}.$$

With these conditions, and using (7), the (constant) scalar curvature is given by

$$\tau = 6\mathcal{B},$$

and hence (19) reduces to

(45)
$$a = x_1^2 \tfrac{\tau}{6} + x_1 P + x_2 Q + \xi,$$
$$b = x_2^2 \tfrac{\tau}{6} + x_1 S + x_2 T + \eta,$$
$$c = x_1 x_2 \tfrac{\tau}{6} + x_1 U + x_2 V + \gamma.$$

Now the two cases are obtained just observing that the last three equations in (43) transform into

$$\tfrac{\tau}{6}\xi - \{Q(T - U) + V(P - V) - 2(Q_4 - V_3)\} = 0,$$

$$\tfrac{\tau}{6}\gamma - \{-QS + UV + T_3 - U_3 + P_4 - V_4\} = 0,$$

$$\tfrac{\tau}{6}\eta - \{S(P - V) + U(T - U) - 2(S_3 - U_4)\} = 0,$$

and noting that when the scalar curvature τ does not vanish we can determine ξ, η and γ from above, while if $\tau = 0$ then we get (41).

Finally, the eigenvalues and the minimal polynomial of the Jacobi operators for the two cases are obtained as a direct application of Remark 1, since the

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eigenvalues and the minimal polynomial of the self-dual Weyl tensor W^+ determine the behavior of the Jacobi operators of a pointwise Osserman self-dual manifold. $\square$

As a consequence of the previous theorem and Remark 1, we have the following characterization of Jordan-Osserman Walker metrics.

Theorem 5 *Let (M, g) be a Jordan-Osserman Walker 4-manifold. Then one of the following holds:*

   (i) *If the Jacobi operators are diagonalizable, then (M, g) is either flat or locally isometric to a paracomplex space form.*

  (ii) *If the Jacobi operators are non diagonalizable, then either*

     (ii.1) *the Jacobi operators are two-step or three-step nilpotent*

     (ii.2) *the metric g is given by* (1)–(2).

Proof. Four-dimensional Jordan-Osserman manifolds with diagonalizable Jacobi operators have been classified at [3], showing that they correspond to real, complex or paracomplex space forms. Next note that real and complex space forms do not support a Walker metric unless they are flat. Indeed, any space of constant curvature is locally conformally flat and hence vanishing of W^+ shows that any such a Walker metric must be flat. Analogously, Kähler metrics of constant holomorphic sectional curvature have zero Bochner tensor, which shows that $W^+ = 0$ [6], [14]. Hence, no Kähler metric of constant holomorphic sectional curvature may be Walker unless it is flat.

On the other hand, if the Jacobi operators are nondiagonalizable, they must be of Type II or III, since Type Ib cannot occur [3]. Then, since anti-self-dual Jordan-Osserman Walker metrics have vanishing scalar curvature, the corresponding Jacobi operators are either two-step or three-step nilpotent. The only remaining case is that of self-dual Jordan-Osserman Walker metrics, which corresponds to the Main Theorem, thus finishing the proof. $\square$

Remark 6 Para-Kähler manifolds of constant paraholomorphic sectional curvature α can be easily described as Walker manifolds. For instance, let a, b and c be the coordinate functions given by

$$a(x_1, x_2, x_3, x_4) = \alpha x_1^2, \quad b(x_1, x_2, x_3, x_4) = \alpha x_2^2, \quad c(x_1, x_2, x_3, x_4) = \alpha x_1 x_2,$$

and J the paracomplex structure

$$J\partial_1 = -\partial_1, \quad J\partial_2 = -\partial_2, \quad J\partial_3 = -a\partial_1 - c\partial_2 + \partial_3, \quad J\partial_4 = -c\partial_1 - b\partial_2 + \partial_4.$$

Then $(\mathbb{R}^4, g, J)$ is a para-Kähler manifold of constant paraholomorphic sectional curvature α.

Remark 7 Notice from Remark 1 that any anti-self-dual Jordan-Osserman Walker metric has necessarily nilpotent Jacobi operators. However besides the fact that many nilpotent Jordan-Osserman metrics are known, none of the previous examples are anti-self-dual but all of them correspond to special cases of Theorem 4. The general expression of W_{11}^+ at (15) makes it quite untractable and hence it is very difficult to obtain the general form of anti-self-dual Walker metrics. However, for the special choice of $a(x_1, x_2, x_3, x_4) = b(x_1, x_2, x_3, x_4) = c(x_1, x_2, x_3, x_4)$, anti-self-dual Einstein metrics are characterized by

$$a_{11} = a_{22} = -a_{12},$$
$$a_{13} = a_{14},$$
$$a_{23} = a_{24},$$
$$a_{33} + a_{44} = 2a_{34}.$$

Now, it follows that $a(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - 2x_1x_2$ defines an Osserman anti-self-dual Walker metric with two-step nilpotent Jacobi operators.

Remark 8 Note that any Type III Jordan-Osserman Walker metric is Ricci flat, and thus the Jacobi operators are three-step nilpotent. The existence of non-Ricci flat Type III metrics is still an open problem.

Remark 9 As a final remark, recall that when dealing with Osserman metrics, main attention is usually paid to the behavior of the eigenvalues of the Jacobi operators. However when the metric under consideration is of indefinite signature, the corresponding eigenspaces play a basic role too. Indeed, four-dimensional complex and paracomplex space forms have diagonalizable Jacobi operators with eigenvalues $\{\alpha, \frac{\alpha}{4}, \frac{\alpha}{4}\}$ but the eigenspace corresponding to the multiple eigenvalue $\frac{\alpha}{4}$ inherits a definite (positive or negative) metric in the complex case in opposition to the paracomplex case, where the induced metric has Lorentzian signature. The latter is necessarily the case for any Type II Jordan-Osserman metric.

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Authors' address

Department of Geometry and Topology, Faculty of Mathematics,
University of Santiago de Compostela,
15782 Santiago de Compostela, Spain.

E-mails: xtjosec@usc.es   xtedugr@usc.es   ravazlor@usc.es